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Do Fuzzy Quantum Structures Exist?

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Recently, some fuzzy quantum structures were introduced. We focus on the fuzzy quantum logics arising from the isomorphism of some quantum logics and some systems of fuzzy subsets of the ordering sets of states. In general, a fuzzy quantum logic is equipped with the pointwise-defined fuzzy connectives generated by a common generator **g**. Stressing the pointwise nature of fuzzy structures and omitting the global properties of quantum elements, we find that only crisp values of elements of a fuzzy quantum logic are allowed. Consequently, fuzzy quantum structures do not exist! However, there exist quantum structures of fuzzy subsets.

1. INTRODUCTION

Fuzzy sets in the framework of quantum mechanics were introduced by several authors (e.g., Aerts *et al.*, 1992; Dvurečenskij and Riečan, 1991; Pykacz, 1987). The pointwise nature of fuzzy sets and the fuzzy connectives does not fit the global nature of the quantum structures, e.g., of the quantum logics and the quantum connectives. This is, e.g., the case of fuzzy quantum spaces of Dvurečenskij and Riečan (1991), whose elements are, up to an isomorphism (Navara, n.d.), quasi crisp sets of Aerts *et al.* (1992) (i.e., their range is $\{0, 1/2, 1\}$, but the probability of values 1/2 is zero).

Recently, Pykacz (1992, 1994) introduced the notion of a (generalized) fuzzy quantum logic using the notation of fuzzy set theory only. Pykacz's approach was based on the Maczyński (1974) theorem proving that any quantum logic $\mathcal{L}$ with an ordering set of states $\mathcal{S}$ can be isomorphically represented in the form of a special family $\mathcal{L}(\mathcal{S})$ of fuzzy subsets of $\mathcal{S}$; see Section 2. In Section 3 we discuss the case of a general system of fuzzy subsets of some universe X equipped with pointwise-defined fuzzy connectives, as a quantum logic in the traditional sense (Beltrametti and Cassinelli, 1981). Finally, we consider fuzzy quantum logics from the fuzzy set point of view.

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This means that we require that a restriction of a fuzzy structure (hence of a fuzzy quantum logic, too) to a subuniverse Y of X should preserve the type of the underlying structure.

2. FUZZY QUANTUM LOGICS

Let $\mathcal{L}$ be a quantum logic (Beltrametti and Cassinelli, 1981), i.e., an orthocomplemented σ-orthocomplete orthomodular poset, and let $\mathcal{S}$ be any nonempty system of states (σ-additive probability measures) on $\mathcal{L}$ (e.g., Maczyński, 1974). For an element $\mathbf{a} \in \mathcal{L}$ we introduce a fuzzy subset $\phi(\mathbf{a}) = A$ of $\mathcal{S}$ putting

$$A(\mathbf{s}) = \mathbf{s}(\mathbf{a}), \qquad \mathbf{s} \in \mathcal{S} \tag{1}$$

Recall that a fuzzy subset F of $\mathcal{S}$ is a mapping $F: \mathcal{S} \to [0, 1]$, and that the system $\mathcal{F}(\mathcal{S})$ of all fuzzy subsets of $\mathcal{S}$ can be treated as a power set, $\mathcal{F}(\mathcal{S}) = [0, 1]^{\mathcal{S}}$ (Zadeh, 1965).

It is easy to show that the system $\mathcal{L}(\mathcal{S}) = \{A \in \mathcal{F}(\mathcal{S}); \exists \mathbf{a} \in \mathcal{L}, A = \phi(\mathbf{a})\}$ fulfills the following two properties:

(F1) $\mathcal{L}(\mathcal{S})$ contains the smallest element of $\mathcal{F}(\mathcal{S})$, $\mathbf{0}_{\mathcal{S}} = \phi(\mathbf{0}) \in \mathcal{L}(\mathcal{S})$.

(F2) $\mathcal{L}(\mathcal{S})$ is closed under the standard fuzzy complementation (Zadeh, 1965)

$$A = \phi(\mathbf{a}) \in \mathcal{L}(S) \Rightarrow A' = \mathbf{1}_{\mathcal{S}} - A = \phi(\mathbf{a}^{\perp}) \in \mathcal{L}(\mathcal{S})$$

Further, suppose that $\mathcal{S}$ is an ordering set of states on $\mathcal{L}$ (Maczyński, 1974), i.e., if for two elements $\mathbf{a}, \mathbf{b} \in \mathcal{L}$ one has $\mathbf{s}(\mathbf{a}) \leq \mathbf{s}(\mathbf{b})$ for all $\mathbf{s} \in \mathcal{S}$, then $\mathbf{a} \leq \mathbf{b}$. Then the next two properties are fulfilled in $\mathcal{L}(\mathcal{S})$:

(F3) $A \in \mathcal{L}(\mathcal{S}), A \leq A' \Rightarrow A = \mathbf{0}_{\mathcal{S}}$.

(F4) $\{A_n\} \subset \mathcal{L}(\mathcal{S}), A_n \cap A_m = \mathbf{0}_{\mathcal{S}}$ whenever $n \neq m \Rightarrow \cup A_n \in \mathcal{L}(\mathcal{S})$, where $\cap$ and $\cup$ are the bold fuzzy connectives of Giles (1976).

Note that the usual order on [0, 1] induces the partial order on fuzzy subsets of $\mathcal{S}$, i.e., $A \leq A'$ iff $A(\mathbf{s}) \leq A'(\mathbf{s})$ for all $\mathbf{s} \in \mathcal{S}$. Let $A = \phi(\mathbf{a})$. Then $A \leq A'$ is equivalent to $\mathbf{s}(\mathbf{a}) \leq \mathbf{s}(\mathbf{a}^{\perp})$, which implies $\mathbf{a} \leq \mathbf{a}^{\perp}$. But then $\mathbf{0} = \mathbf{a} \wedge \mathbf{a}^{\perp} = \mathbf{a}$ and consequently $\mathbf{A} = \phi(\mathbf{0}) = \mathbf{0}_{\mathcal{S}}$.

Further, (F4) corresponds to the σ-orthocompleteness of $\mathcal{L}$. Let $\{A_n\} = \{\phi(\mathbf{a}_n)\}$. The sequence $\{\mathbf{a}_n\}$ is mutually orthogonal if and only if for all $n \neq m$ one has $\mathbf{s}(\mathbf{a}_n) \leq \mathbf{s}(\mathbf{a}_m^{\perp}) = 1 - \mathbf{s}(\mathbf{a}_m)$ for each $\mathbf{s} \in \mathcal{S}$, i.e., iff $n \neq m$ implies $A_n \cap A_m = \max(0, A_n + A_m - 1) = \mathbf{0}_{\mathcal{S}}$, where $\cap$ is the bold intersection of fuzzy sets introduced by Giles (1976). The σ-orthocompleteness of $\mathcal{L}$ ensures $\mathbf{a} = (\vee\, \mathbf{a}_n) \in \mathcal{L}$ and $\mathbf{s}(\mathbf{a}) = \Sigma\, \mathbf{s}(\mathbf{a}_n) \leq 1$ for each $\mathbf{s} \in \mathcal{S}$. But then $A = \phi(\mathbf{a}) \in \mathcal{L}(S)$ and $A = \Sigma\, \phi(\mathbf{a}_n) = \Sigma\, A_n = \min(1, \Sigma\, A_n) = \cup\, A_n$, where $\cup$ is the bold union of fuzzy sets introduced by Giles (1976).

Conversely, a system $\mathcal{U} \subset \mathcal{F}(\mathsf{X})$ of fuzzy subsets of given universe X fulfilling (F1)–(F4) was called by Pykacz (1994) a *fuzzy quantum logic*. Note that the earlier notion of a fuzzy quantum logic (Pykacz, 1992) was defined by means of an algebraic sum, which is not a fuzzy connective. This was a consequence of a direct application of Maczyński's theorem, which was formulated for systems of functions (and not for fuzzy subsets).

Remember that the intrinsic partial order on $\mathcal{L}(\mathcal{S})$ induces the join and the meet on $\mathcal{L}(\mathcal{S})$. If we want to replace $\vee$ and $\wedge$ by $\cup$ and $\cap$ and to preserve the properties of quantum logics, we have to show the coincidence $\vee = \cup$ and $\wedge = \cap$ on $\mathcal{L}(\mathcal{S})$. Note that the join $\vee$ is not the usual Zadeh (1965) fuzzy union (which is defined as a pointwise maximum), but it is the least upper bound in $\mathcal{L}(\mathcal{S})$. The main result of Pykacz (1994) solves the above problem.

Theorem 1. Each fuzzy quantum logic $\mathcal{U}$ is a quantum logic in the traditional sense, i.e., $\mathcal{U} \subset \mathcal{F}(\mathsf{X})$ fulfilling (F1)–(F4) is an orthocomplemented σ-orthocomplete orthomodular poset with respect to the standard fuzzy set partial order, standard fuzzy complement (as orthocomplement), and bold fuzzy connectives (as the join and the meet). ■

3. g-FUZZY QUANTUM LOGICS

Pykacz's results have led to the following problem: which systems of fuzzy subsets of a given universe X equipped with fuzzy connectives of complement, union, and intersection can be treated as quantum logics? Fuzzy connectives are built up pointwise by means of some operations on the unit interval. Namely, the fuzzy complement A^c arises from a unary operation $\mathbf{c}$ which is an order-reversing involution on [0, 1],

$$A^c(x) = \mathbf{c}(x), \qquad x \in \mathsf{X} \tag{2}$$

By Trillas (1979), for each such $\mathbf{c}$ there is a generator $\mathbf{g}$ (not unique!), $\mathbf{g}$: $[0, 1] \to [0, 1]$ is an increasing bijection, so that

$$\mathbf{c}(u) = \mathbf{g}^{-1}(1 - \mathbf{g}(u)), \qquad u \in [0, 1] \tag{3}$$

The fuzzy union $\cup$ is induced by a t-conorm $\mathbf{S}$ and the fuzzy intersection $\cap$ is induced by a t-norm $\mathbf{T}$,

$$\begin{aligned} (A \cup B)(x) &= \mathbf{S}(A(x), B(x)), \\ (A \cap B)(x) &= \mathbf{T}(A(x), B(x)), \qquad x \in \mathsf{X} \end{aligned} \tag{4}$$

Triangular norms and conorms are associative, commutative, nondecreasing binary operations on [0, 1] with unit element 1 or 0, respectively. For more details see Schweizer and Sklar (1983). **S** and **T** are supposed to fulfill the DeMorgan laws, i.e.,

$$\mathbf{S}(u, v) = \mathbf{c}(\mathbf{T}(\mathbf{c}(u), \mathbf{c}(v))), \qquad u, v \in [0, 1] \tag{5}$$

Examining the axioms of a traditional quantum logic taking the fuzzy complement as the orthocomplement, the join as the fuzzy union, and the meet as the fuzzy intersection, we come to the following result (Mesiar, 1993b):

There is a generator **g** such that for all $u, v \in [0, 1]$ one has

$$\begin{aligned} \mathbf{c}(u) &= \mathbf{g}^{-1}(1 - g(\mathbf{u})) \\ \mathbf{S}(u, v) &= \mathbf{g}^{-1}(\min(1, \mathbf{g}(u) + \mathbf{g}(v))) \\ \mathbf{T}(u, v) &= \mathbf{g}^{-1}(\max(\mathbf{g}(u) + \mathbf{g}(v) - 1, 0)) \end{aligned} \tag{6}$$

Recall that if $\mathbf{g}(u) = u$ is the identity on [0, 1], we get just the basis of Pykacz's approach.

Of course, in general the fuzzy union $\cup$ induced by **S** need not coincide with the usual join $\vee$ induced by the intrinsic partial order of fuzzy subsets. This is ensured only if no nontrivial weak empty set is contained in the system we are dealing with, i.e., if $A \leq A^c$ implies $A = \mathbf{0}_X$. Recall that $A \leq A^c$ is equivalent to $A(x) \leq \mathbf{g}^{-1}(1/2)$ for all $x \in \mathrm{X}$. All the above facts are summarized in the following theorem. For a detailed proof see Mesiar (1994).

Theorem 2. A system $\mathcal{V} \subset \mathcal{F}(\mathrm{X})$ of fuzzy subsets of X equipped with pointwise-defined fuzzy connectives is a quantum logic in the traditional sense if and only if **c**, **S**, and **T** are generated by a common generator **g** as in (6) and the following four properties are fulfilled:

(GF1) $\mathbf{0}_X \in \mathcal{V}$.
(GF2) $A \in \mathcal{V} \Rightarrow A^c \in \mathcal{V}$.
(GF3) $A \in \mathcal{V}, A \leq A^c \Rightarrow A = \mathbf{0}_X$.
(GF4) $\{A_n\} \subset \mathcal{V}, A_n \cap A_m = \mathbf{0}_X$ whenever $n \neq m \Rightarrow \cup A_n \in \mathcal{V}$. ∎

The system $\mathcal{V}$ fulfilling (GF1)–(GF4) with fuzzy connectives generated by a generator **g** is called a **g**-*fuzzy quantum logic*. Recall that even here we can apply a generalized Mączyński theorem replacing the additivity of a state by a pseudoadditivity (Mesiar, 1993a), i.e.,

$$\mathbf{a} \perp \mathbf{b} \Rightarrow \mathbf{s}(\mathbf{a} \vee \mathbf{b}) = \mathbf{S}(\mathbf{s}(\mathbf{a}), \mathbf{s}(\mathbf{b})) = \mathbf{g}^{-1}(\min(1, \mathbf{g}(\mathbf{s}(\mathbf{a})) + \mathbf{g}(\mathbf{s}(\mathbf{b}))))$$

The above results show that, up to an isomorphism given by a generator **g**, Pykacz's results cover the problem of quantum logics of fuzzy sets.

4. FUZZY QUANTUM LOGICS "DO NOT EXIST"

The nature of fuzzy structures is pointwise. Take an arbitrary nonempty crisp subset $\mathsf{Y} \subset \mathsf{X}$. Then the restriction of a given fuzzy structure on X to Y is again a fuzzy structure of the same type (e.g., fuzzy σ-algebras, **T**-tribes, generated tribes, etc.). Looking at a fuzzy quantum logic $\mathcal{V}$ from this point of view (i.e., neglecting the global properties of underlying elements) leads us to the following conclusion: for each $x \in \mathsf{X}$, the restriction $\mathcal{V}(x)$ of $\mathcal{V}$ to the singleton $\mathsf{Y} = \{x\}$ should be again a fuzzy quantum logic. But then the elements of $\mathcal{V}(x)$ are real numbers from the unit interval, i.e., $\mathcal{V}(x)$ is linearly ordered and due to the condition (F3), the only element $u \in \mathcal{V}(x)$ such that $u \leq u'$, i.e., $u \leq 1/2$, is $u = 0$. Consequently $\mathcal{V}(x) = \{0, 1\}$ for all $x \in \mathsf{X}$, which means that for each $A \in \mathcal{V}$ and $x \in \mathsf{X}$ one has $A(x) \in \{0, 1\}$. But then a fuzzy quantum logic contains only crisp elements! In other words, no fuzzy quantum logic exists. Of course, this is only speaking in a certain philosophical sense. From the physical point of view, the restriction of a fuzzy quantum logic defined on a universe X to a subuniverse Y corresponds to the restriction of the state space and hence there is no physical reason to preserve the quantum properties (e.g., that Y is an ordering set of states). Hence the above-investigated structures of fuzzy sets have a real meaning and the only conclusion of our considerations is the following one: do not use the notion "fuzzy quantum structures," such as fuzzy quantum logics, but use preferably the notion "quantum structures of fuzzy sets," e.g., the quantum logics of fuzzy sets.

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